We may apply similar arguments in the case of the indecomposable constituents $\mathfrak{V}_\sigma$ of the second regular representations $\mathfrak{S}$ of $J/\mathfrak{p}$. We have formulas analogous to (9) and (11). Comparing them with (9) and (11), we find

$$(\mathfrak{V}_\sigma) \longleftrightarrow \sum_{\rho=1}^{l} d_{\rho\sigma}\,\mathfrak{T}_\rho \longleftrightarrow (\mathfrak{U}_\sigma). \tag{13}$$

If A is normal and simple, we have only one $\mathfrak{T}_\lambda$, $l = 1$, and the corresponding q_λ gives the square of the $\mathfrak{p}$-index of A.[5]

In the case of a semisimple algebra A, we can always replace K by an algebraic extension field such that with regard to this new ground field, the numbers q_ρ and r_σ are to be replaced by 1. With regard to such a ground field, we have

$$(\mathfrak{U}_\sigma) \longleftrightarrow (\mathfrak{V}_\sigma) \longleftrightarrow \sum_{\rho} d_{\rho\sigma}\mathfrak{T}_\rho;\ \mathfrak{T}_\rho \longleftrightarrow \sum_{\sigma} d_{\rho\sigma}\,\mathfrak{F}_\sigma, \tag{14}$$

$$c_{\kappa\lambda} = \sum_{\rho} d_{\rho\kappa} d_{\rho\lambda}. \tag{15}$$

[1] For the properties of the regular representations and intertwining matrices cf. R. Brauer–C. Nesbitt, these PROCEEDINGS **23**, 236 (1937) (referred to under B. N.). T. Nakayama, *Ann. of Math.* (2), **39**, 361 (1938); C. Nesbitt, *Ann. of Math.* (2), **39**, 634 (1938); and two forthcoming papers by R. Brauer.

[2] See L. E. Dickson, *Algebras and Their Arithmetics*, Chicago, 1923, §42.

[3] Cf. for instance, A. A. Albert, *Modern Higher Algebra*, Chicago, 1937, p. 296.

[4] W. Burnside, *Proc. London Math. Soc.* (2), **7**, 8 (1909).

[5] H. Hasse, *Math. Ann.*, **107**, 731 (1933).

NUMBER OF THE SUBGROUPS OF ANY GIVEN ABELIAN GROUP

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Every abelian group whose order is not a power of a prime number is known to be the direct product of its Sylow subgroups. Hence the number of its subgroups is the product of the numbers of the subgroups contained in its Sylow subgroups. It is therefore easy to determine the total number of the subgroups of any given abelian group whenever we know the numbers of the subgroups contained in its Sylow subgroups. Hence we shall confine our attention for the present to a determination of the total number of the subgroups contained in a given abelian group G of order p^m, p

being a prime number. It is known that every possible set of independent generators of G involves the same number of operators. Let $s_1, s_2, \ldots, s_k$ be such a set of independent generators of G, arranged in ascending order of magnitude if the orders of two such successive generators are not all equal to each other.

We shall first determine the total number of the cyclic subgroups of an arbitrary order p^α contained in G. For this purpose it is only necessary to find the total number of the operators of G whose orders divide p^α and to subtract from this number the total number of the operators of G whose orders are less than p^α. The remainder thus obtained divided by $p^\alpha - p^{\alpha-1}$ gives the total number of the cyclic subgroups of order p^α contained in G since each of these cyclic subgroups contains exactly $p^\alpha - p^{\alpha-1}$ operators of order p^α which do not appear in any other cyclic group of order p^α, because these operators separately generate the cyclic group of this order in which they appear.

The number of the operators contained in G whose orders divide p^α is the product of k numbers, r of these are equal to p^α and represent respectively orders of operators generated by the given set of independent generators of G whose orders are separately at least equal to p^α, while the remaining $k - r$ are respectively equal to the orders of the operators in the given set of independent generators which are separately less than p^α. The number of the operators contained in G whose orders are less than p^α is also equal to the product of k numbers. The last $k - r$ of these may be so selected that they are the same as in the preceding case, while the remaining r of them are separately equal to $p^{\alpha-1}$ instead of p^α. Hence there results the following theorem: *If an abelian group of order* p^m, p *being a prime number, has a set of* k *independent generators which involves* r *operators which are separately at least equal to* p^α *while the orders of the rest of these operators are separately less than* p^α, *the number of its cyclic subgroups of order* p^α *may be obtained by dividing by* $p - 1$ *the product of the latter orders and* $(p^r - 1)p^{(\alpha-1)(r-1)}$.

The number of the operators of order p^α contained in G is always larger than the number of its cyclic subgroups of this order except when $p = 2$ and $\alpha = 1$. In this special case these two numbers are equal to each other. If the order of an abelian group is not a power of a prime number its cyclic subgroups are the direct products of the cyclic subgroups contained in its Sylow subgroups. Hence the number of its cyclic subgroups is then the product of the numbers of its cyclic subgroups contained in its Sylow subgroups. It may be noted here that in the *Encyclopaedia Britannica* (1938) there appears a statement under the entry of the term "Groups," volume 10, page 914, which implies that a set of independent generators of an arbitrary abelian group can be so selected that each of them generates a Sylow subgroup. This is evidently only possible when the given abelian group is cyclic. Various other incorrect statements appear under the same entry in this encyclopedia.

It is considerably more difficult to determine the number of the non-cyclic subgroups of G than to find a formula which gives the number of the cyclic subgroups contained therein. A few general theorems simplify the determination of the former number. One of these may be stated as follows: An abelian group of order p^m contains one and only one subgroup which satisfies the two conditions that it has a set of independent generators involving as many operators as appear in a set of independent generators of G and that the orders of the former independent generators are all equal to a given power of p which does not exceed the order of the smallest independent generator of G. The number of these subgroups contained in G is therefore equal to the index of the power of p which is equal to the order of the smallest independent generator of G. Each of these subgroups is a characteristic subgroup of G since it is composed of all the operators of G whose orders divide a given number.

Another general theorem relating to the number of the subgroups of G may be stated as follows: If all the orders of a set of independent generators of a subgroup of G are equal to the orders of a set of independent generators of G with the exception of the generator of highest order in the latter set, and the order of the largest independent generator of the subgroup is at least equal to the order of the next to the largest independent generator of G, then there is one and only one such subgroup in G for every order of the last independent generator of the subgroup which does not exceed the order of the largest independent generator of G. The number of these subgroups of G is therefore equal to one more than the difference between the indexes of the power of p which is equal to the order of the largest independent generator of G and the power of p which is equal to the order of the next to the largest operator in the given set of the independent generators of G.

Since the number of the operators in a set of independent generators of a subgroup of G is equal to the index of the power of p which is equal to the order of the group generated by all the operators of order p contained in this subgroup it results that a subgroup of G cannot have a set of independent generators which involves more operators than appear in every set of independent generators of G. In particular, a necessary and sufficient condition that a subgroup of G has a set of as many independent generators as G itself has, is that it involves all the operators of order p contained in G. To determine the number of the subgroups of a given type which appear in G it is only necessary to determine the number of ways in which a set of independent generators of a subgroup of this type can be selected from the operators of the group and to divide this number by the number of ways in which a set of independent generators of such a subgroup can be selected from its own operators. In particular, the number of the subgroups of G which have $k_1 \leqq k$ independent generators and involve no operator of order p^2 is known to be

$$\frac{(p^k - 1)(p^k - p)\ldots(p^k - k_1^{k-1})}{(k_1^k - 1)(k_1^k - p)\ldots(k_1^k - k_1^{k-1})}.$$

In determining the number of the subgroups of a given type contained in G it is only necessary to consider the operators of G whose orders do not exceed the order of the largest operator in a set of independent generators of such a subgroup. It is convenient to consider first the number of ways in which an independent generator of highest order of such a subgroup can be selected from the operators of G and then to consider the number of ways in which the independent generators of the successively lower orders in a set of independent generators of the subgroup can be selected from the operators of G and to divide the product of these numbers by the number of ways in which the operators of a set of independent generators of the subgroup can be selected from the operators of the subgroup. In each particular case it is only necessary to consider the operators of the group, or of the subgroup, whose order is equal to the order of the independent generator in question.

To illustrate this general method we proceed to consider the total number of the subgroups of G when $k = 2$ and the orders of s_1 and s_2 are p^β and p^γ, respectively. When the smaller of the two generators of such a subgroup is p the larger independent generator can be selected from the operators of G in $p^{\beta-1}$ times as many ways as from the operators of the subgroup while the smaller of the two independent generators can then be selected from the operators of G in the same number of ways as from the operators of the subgroup in question provided the order of the larger independent generator of the subgroup is at least p^β. As the order of the larger independent can be chosen in $\gamma - \beta$ ways so as to exceed p^β the number of these subgroups in which one independent generator is of order p is $(\gamma - \beta)p^{\beta-1}$. As similar remarks apply to the cases when the smaller independent generators of the subgroup in question have various larger values up to p^β, the number of the subgroups in which the larger independent generator is at least as large as p^β is $(\gamma - \beta)(p^\beta - 1)/p - 1$.

We proceed to consider the case when neither of the two generators of the subgroups in question has an order which exceeds p^β. If these generators have the same order there is clearly one and only one subgroup since such a subgroup is then composed of all the operators of G whose orders divide the common order of the generators. It remains to consider the case when the two independent generators of the subgroups in question have different orders, say p^{α_1}, p^{α_2}, $\alpha_2 > \alpha_1$. The number of the operators of order p^{α_2} contained in G is then $p^{2\alpha_2} - p^{2x_2-2}$ and the number of the operators of this order contained in such a subgroup is $p^{\alpha_1+\alpha_2} - p^{\alpha_1+\alpha_2-1}$. The number of these subgroups is therefore the quotient of the given numbers of operators since the numbers of ways in which the smaller independent generator can be chosen after the larger independent generator has been selected is the same for the group as for the subgroup.

Hence the number of the subgroups in which the two independent generators are of orders p^{α_1} and p^{α_2} is $p + 1$ times $p^{\alpha_2-\alpha_1-1}$. The exponent $\alpha_2 - \alpha_1 - 1$ cannot exceed $\beta - 2$ and has this value only when $\alpha_2 = \beta$ and $\alpha_1 = 1$. There are two cases in which this exponent is $\beta - 3$, viz., when $\alpha_2 = \beta$ and $\alpha_1 = 2$ or when $\alpha_2 = \beta - 1$ and $\alpha_1 = 1$. As similar considerations apply to the other cases the number of these subgroups is obviously given by the formula

$$(p + 1)(p^{\beta-2} + 2p^{\beta-3} + \ldots\ldots + (\beta - 1)p^0$$

whenever $\beta > 1$. In this case the total number of the subgroups in the abelian group of order p^m whose independent generators are of orders p^β, p^γ is therefore given by the formula

$$\beta + (p + 1)(p^{\beta-2} + 2p^{\beta-3} + \ldots + (\beta - 1)p^0) + (\gamma - \beta)(p^\beta - 1)/(p - 1).$$

The entire group is included among the subgroups in the enumeration by this formula. When $\beta = 1$ the first and the last term of this formula, taken together, give the total number of the subgroups in question.

THE TRANSFORMATION OF A LAGRANGIAN SERIES INTO A NEWTONIAN SERIES

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A series of type

$$\sum_{n=o}^{\infty} a_n \frac{(z - c - 1)(z - c - 2) \ldots (z - c - n)}{(z + c + 1)(z + c + 2) \ldots (z + c + n)} = \sum a_n R_n(z) \quad (1)$$

will be called a Lagrangian series on account of the extensive investigations of René Lagrange[1] relating to the expansion of functions in series of this and allied types.

A transformation of a Newtonian series into a series of type (1) may be effected with the aid of Lagrange's expansion